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LETTER TO THE EDITOR

A note on (p, q)-oscillators and bibasic hypergeometric functions

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Abstract. We examine the relation between the representation theory of a two-parameter deformation of the oscillator algebra and certain bibasic Laguerre functions and polynomials.

As part of a general study of the application of quantum algebras to the theory of special functions, we have established [1] the relation between the q-oscillator algebra and q-Laguerre functions and polynomials [2, 3]. A two-parameter deformation of the oscillator algebra has been presented by several authors recently [4]. We here investigate the connection between this (p, q)-algebra and bibasic extensions of special functions [3].

The (p, q)-oscillator algebra is generated by three elements A, $A^\dagger$ and N obeying the relations [4]

$$\begin{aligned} &[N, A] = -A \qquad [N, A^\dagger] = A^\dagger \\ &AA^\dagger - qA^\dagger A = p^{-N} \qquad AA^\dagger - \frac{1}{p}A^\dagger A = q^N. \end{aligned} \tag{1}$$

In the limit $p \to 1^+$, these reduce to the defining relations of the q-oscillator algebra used in [1]; for $p = q$, one recovers the relations introduced in [5].

Let us point out from the outset that the (p, q)-algebra (1) can be mapped on the one-parameter deformation of the oscillator algebra. Indeed, if instead of A and $A^\dagger$ one takes

$$a = p^{N/2}A \qquad a^\dagger = p^{(N-1)/2}A^\dagger \tag{2}$$

as generators, the relations (1) become

$$\begin{aligned} &[N, a] = -a \qquad [N, a^\dagger] = a^\dagger \\ &aa^\dagger - (qp)a^\dagger a = 1 \qquad aa^\dagger - a^\dagger a = (qp)^N \end{aligned} \tag{3}$$

where only the combination qp appears as a parameter.

The connection between quantum alegbras and special functions depends not only on the particular algebra which is considered but also on the special realization which is used. In the present context, although the (p, q)-oscillator algebra does not really involve two

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independent parameters, the presentation (1) still proves useful to construct models for bibasic special functions.

Consider the following realization of the algebra (1) on the space of all finite linear combinations of the monomonials z^n, $z \in \boldsymbol{C}$, $n \in \boldsymbol{Z}$:

$$A = \frac{1}{z}\frac{T_p^{-1}p^{-\rho} - T_q q^{\rho}}{(p^{-1} - q)} \qquad A^{\dagger} = z \qquad N = \rho + z\frac{\mathrm{d}}{\mathrm{d}z} \tag{4}$$

where ρ is a non-integer complex parameter and T_q stands for the q-shift operator that acts according to $T_q f(z) = f(qz)$ on functions of z.

The action of the generators on the basis vectors $f_m = z^n$, where $m = \rho + n$, is given by

$$Af_m = \frac{(p^{-m} - q^m)}{(p^{-1} - q)} f_{m-1} \qquad A^{\dagger} f_m = f_{m+1} \qquad Nf_m = mf_m. \tag{5}$$

To proceed in analogy with [4], we shall need a (p, q)-analogue of the exponential. A convenient definition is

$$E_{p,q}(z) = \sum_{n=0}^{\infty} \left(\frac{q}{p}\right)^{n(n-1)/2} \frac{z^n}{[p, q; p, q]_n} \qquad |q| < 1, |pq| < 1 \tag{6}$$

where

$$[p^{\mu}, q^{\nu}; p, q]_n = \left(\frac{1}{p^{\mu}} - q^{\nu}\right)\left(\frac{1}{p^{\mu+1}} - q^{\nu+1}\right)\cdots\left(\frac{1}{p^{\mu+n-1}} - q^{\nu+n-1}\right). \tag{7}$$

In terms of the q-shifted factorial [3] $(a; q)_n = (1 - a)(1 - aq)\ldots(1 - aq^{n-1})$ one has

$$[p^{\mu}, q^{\nu}; p, q]_n = p^{-(n(n-1)-\mu n)/2}(p^{\mu}q^{\nu}; pq)_n. \tag{8}$$

For arbitrary complex α we can take

$$\begin{aligned} [p^{\mu}, q^{\nu}; p, q]_{\alpha} &= \frac{[p^{\mu}, q^{\nu}; p, q]_{\infty}}{[p^{\mu+\alpha}, q^{\nu+\alpha}; p, q]_{\infty}} \qquad |q| < 1, |pq| < 1 \\ [p^{\mu}, q^{\nu}; p, q]_{\infty} &= \prod_{k=0}^{\infty}\left(\frac{1}{p^{\mu+k}} - q^{\nu+k}\right). \end{aligned} \tag{9}$$

In the limit $p \to 1$, one recovers from (6) the standard q-analogue of the exponentials that was used in [1].

A natural (p, q)-generalization of the derivative is $(T_p - T_q^{-1})/z$; it acts as follows on $E_{p,q}(z)$:

$$\frac{1}{z}(T_p - T_q^{-1})E_{p,q}(z) = -\frac{p}{q}E_{p,q}(z). \tag{10}$$

The function $E_{p,q}(z)$ can be expressed as a bibasic hypergeometric series; these are defined in general by [3]:

$$\begin{aligned} \Phi\left[\begin{matrix} a_1, \ldots, a_n : c_1, \ldots, c_r \\ b_1, \ldots, b_m : d_1, \ldots, d_s \end{matrix}; q, p; z\right] &= \sum_{l=0}^{\infty} \frac{(a_1; q)_l \ldots (a_n; q)_l}{(q; q)_l (b_1; q)_l \ldots (b_m; q)_l} \frac{(c_1; p)_l \ldots (c_r; p)_l}{(d_1; p)_l \ldots (d_s; p)_l} \\ &\quad \times [(-1)^l q^{l(l-1)/2}]^{1+m-n} \cdot [(-1)^l p^{l(l-1)/2}]^{s-r} z^l. \end{aligned} \tag{11}$$

One has

$$E_{p,q}(z) = \Phi\left[\begin{matrix} 0 & : & - \\ - & : & 0 \end{matrix}; pq, q; -pz\right] = \Phi\left[\begin{matrix} - & : & 0 \\ - & : & - \end{matrix}; pq, p; pz\right]. \tag{12}$$

Introduce the operator

$$U(\alpha, \beta) = E_{p,q}(\alpha(p^{-1} - q)A^{\dagger})E_{p,q}(\beta(p^{-1} - q)A). \tag{13}$$

Its matrix elements in the representation (5) are defined by

$$U(\alpha, \beta)z^n = \sum_{k=-\infty}^{\infty} U_{kn}(\alpha, \beta)z^k. \tag{14}$$

Using (6) and identities involving q-shifted factorials, it is straightforward to show that

$$U_{kn}(\alpha, \beta) = \left(\frac{q}{p}\right)^{(n-k)(n-k-1)/2} \beta^{n-k} L_{\rho+k}^{(n-k)}\left(\frac{-\alpha\beta p}{q}; p, q\right) \tag{15}$$

where the bibasic Laguerre function $L_\nu^\lambda(x; p, q)$ is given by

$$L_\nu^{(\lambda)}(x; p, q) = \frac{[p^{\lambda+1}, q^{\lambda+1}; p, q]_\nu}{[p, q; p, q]_\nu} \Phi\left[\begin{matrix} (pq)^{-\nu} & : & 0 \\ (pq)^{\lambda+1} & : & - \end{matrix}; pq, p; (1 - pq)q^{\lambda+\nu+1}x\right]. \tag{16}$$

This provides an algebraic interpretation of a special class of bibasic functions of hypergeometric type. It can be used to obtain generating functions.

From the q-binomial theorem [3] and (8) one has

$$\sum_{n=0}^{\infty} \frac{[p^\mu, q^\nu; p, q]_n}{[p, q; p, q]_n} z^n = \frac{(pq^\nu z; pq)_\infty}{(p^{1-\mu}z; pq)_\infty}. \tag{17}$$

With this formula one shows that

$$U(\alpha, \beta)z^n = E_{p,q}(\alpha(p^{-1} - q)z)z^n \left(\frac{-\beta}{zp^{n+\rho-1}}; pq\right)_{n+\rho}. \tag{18}$$

Inserting this result and the expression (15) of the matrix elements U_{kn} in (14) one obtains after letting $n = 0$, $\beta = -q/p$ and $t = -1/z$

$$E_{p,q}(-\alpha(p^{-1} - q)/t)\left(\frac{-qt}{p^\rho}; pq\right)_\rho = \sum_{k=-\infty}^{\infty} \left(\frac{q}{p}\right)^{k(k+1)/2} t^k L_{\rho-k}^{(k)}(\alpha; p, q). \tag{19}$$

For $p, q \to 1$, this equation reduces to the generating relation [6]

$$e^{-\alpha/t}(1 + t)^\rho = \sum_{k=-\infty}^{\infty} t^k L_{\rho-k}^{(k)}(\alpha) \tag{20}$$

for the ordinary Laguerre functions.

By taking $\rho = 0$ and restricting to analytic functions, one obtains from (5) a representation bounded below. Using $f_m = z^m$ with $m \in Z^+$ as basis vectors, the matrix elements of $U(\alpha, \beta)$ now defined by

$$U(\alpha, \beta)z^n = \sum_{k=0}^{\infty} U_{kn}(\alpha, \beta)z^k \tag{21}$$

are simply obtained by setting $\rho = 0$ in (15). Since for ν integer, $L_\nu^{(\lambda)}(x; p, q)$ is a polynomial of order ν, the matrix elements $U_{kn}(\alpha, \beta)$ are here expressed in terms of (p, q)-Laguerre polynomials. A generating function for these polynomials can also be obtained. It reads

$$E_{p,q}(-\alpha(p^{-1} - q)z)\left(\frac{-q}{zp^n}; pq\right)_n z^n = \sum_{k=0}^{\infty} \left(\frac{q}{p}\right)^{(n-k)(n-k+1)/2} L_k^{(n-k)}(\alpha; p, q)z^k. \tag{22}$$

This is a (p, q)-analogue of the relation [6]

$$e^{-\alpha z}(1 + z)^n = \sum_{k=0}^{\infty} L_k^{(n-k)}(\alpha)z^k \tag{23}$$

for ordinary Laguerre polynomials to which (22) reduces when $p, q \to 1$.

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